Algebra 3 Final Examination
B.Math (Hons.) II Year **Date: November 30th 2009**

Completely justify all answers! Problems 1-5 are worth 8 marks each and each part of problems 6 and 7 is worth 3 marks. Solve problems worth at least 50 marks. Solving more may earn you some extra credit.

Unless otherwise specified, assume the following: All rings are commutative and have 1.

1. a) Let $R = A \times B$, where A and B are rings. Show that the ideals of R are precisely $I \times J$, where I is an ideal of A and J an ideal of B.

b) Which of these ideals of R are maximal? Construct a ring with exactly 2009 maximal ideals.

2. Let M be an R-module with a submodule N. Prove the following:

a) If N and M/N are both finitely generated, then so is M.
b) If N and M/N are both Noetherian, then so is M.

3. Work over the field of complex numbers and answer the following questions.
a) A 6×6 matrix has minimal polynomial $x^3(x+5)$. Give an example of such a matrix and write its characteristic polynomial. Are all such matrices conjugate to each other?

b) Find a polynomial p(x) such that all 2009×2009 matrices with minimal polynomial p(x) are conjugate to teach other and are also diagonalizable .

c) Find a polynomial q(x) such that all 2009×2009 matrices with minimal polynomial q(x) are conjugate to teach other and are NOT diagonalizable.

4. Consider the abelian group G presented by the integer matrix A given below. Recall that this means G is the cokernel of the map between free abelian groups given by A.

$$\begin{pmatrix} 5 & 10 & 100 \\ 15 & 20 & 100 \\ 25 & 30 & 100 \\ 25 & 30 & 100 \end{pmatrix}$$

Diagonalize A. Using this, express G as a direct product of k cyclic groups. What is the maximum possible value of k?

5. Recall that a simple R-module is a nonzero R-module with no nonzero proper submodule. Prove the following statements.

a) A nozero R-module homomorphism between simple R-modules is an isomorphism.

b) Any simple R-module is isomorphic to R/M, where M is a maximal ideal of R.

c) If R is a field, then up to isomorphism there is a unique simple R-module. (Identify this simple module.) Is the converse true?

6. For each of the following, either show that the given element is irreducible in the given ring or factor it into irreducibles.

a) 2009 in the ring of Gaussian integers Z[i].
b) $x^{20} - 2x^{19} + 5x^2 - 5x - 10$ in Q[x].
c) $x^6 + 2x^5 + 4x^4 + 8x^3 + 16x^2 + 32x + 64$ in Q[x].
d) $2009x^4 + 2009x^3 + 2008x^2 + 2008x + 2007$ in Z[x].

7. Answer the following short independent questions.

a) Find a ring with exactly 2009 ideals.

b) Find an integral domain D and element r in D such that rD is maximal among principal ideals of D but is not a maximal ideal of D.

c) For a field F and a vector space V over F, an F-linear map f from V to V is given. Let α be an element adjoined to F to get the ring F[α]. Under what condition(s) can we make V into an F[α]-module by defining $\alpha.v = f(v)$? Briefly but precisely state the necessary and sufficient condition(s).

d) Let A be a square matrix with entries in an algebraically closed field F. Suppose the Jordan form of A consists of just one block and let M be the change of basis matrix that converts A into its Jordan form. Find the matrix that converts A^t into its Jordan form. Why do we assume that F is algebraically closed?